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Finite Möbius Planes of Hering Type I.3 and Related Rank 3 Groups

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INTRODUCTION

If M is a finite Möbius plane of order m and of Hering type I.3, then $G = \mathrm{Aut}(M)$ acts as a rank 3 group of permutations on the points of a distinguished circle k. Further if A is a point of k then G_A has orbits of length $1, 1, m-1$ (on the points of k) and G_A contains a normal subgroup D_A that acts sharply transitively on the long orbit.

In Section 1 we study such permutation groups. Although theorems of Tsuzuku [7] and Hering, Kantor, Seitz (4) may be combined with our Theorem 1 to enumerate all such groups, our application to Möbius planes requires only the relatively elementary argument establishing Theorem 1.

In Section 2 we investigate finite Möbius planes of type I.3 and prove that such a plane must have order 3 or 5. The only such planes are Miquelian [1, p. 273], [6]. Each does admit a collineation group of Hering type I.3.

Thanks are due the referee for correcting the original statement of Theorem 1.

1. ON A CLASS OF IMPRIMITIVE RANK 3 GROUPS

This section is devoted to the proof of the following theorem:

THEOREM 1. *Let G be a transitive group of permutations of $\{1,..., n\}$ having subdegrees $1, 1, n-2$, $n > 6$. Suppose $N_G(G_1)$ has a normal subgroup D_1 that acts sharply transitively on the long G_1-orbit. Then*

(a) *G is the simple group of order 168 and $n = 14$, or*

(b) *D_1 has an Abelian subgroup of odd order and index 2 in D_1 and* $2 \mid |Z(G)|$.

An easy application of the theorem of Hering, Kantor, and Seitz [4] shows that a group satisfying (b) must involve $L_2(q)$ for some odd q. If

$$G = \left\langle SL_2(q), \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\rangle,$$

$$G_1 = \left\langle \begin{pmatrix} a^2 & 0 \\ 0 & a^{-2} \end{pmatrix}, \begin{pmatrix} 1 & b \\ 0 & -1 \end{pmatrix} \;\middle|\; a, b \in GF^*(q) \right\rangle \qquad \text{and} \qquad q \equiv -1 \pmod 4,$$

then G provides an example of the situation described in (b).

Our proof relies on the study of a certain class of involutions, called special involutions. We construct a design I having the special involutions as blocks. If I is degenerate, either $n \leqslant 6$ or case (b) arises. Otherwise we show that I is intimately related to a Steiner triple system. Theorems of M. Hall, Jr. [3], and J. D. King [5] apply to leave only $n = 14$ and G the simple group of order 168.

Suppose G satisfies the hypothesis of the theorem. Then G_1 fixes a second point $i \in \{1,..., n\}$ and $G_1 = G_i$. The set $\{1, i\}$ is a set of imprimitivity, by Wielandt [8, Theorem 7.4]. Thus G is an imprimitive rank 3 group and the underlying set $\{1,..., n\}$ may be relabeled so that the sets of imprimitivity are

$$\mathbf{i} = \{i, i + n/2\} = \mathbf{i} + \mathbf{n}/2, \qquad i = 1,..., n/2.$$

G acts doubly transitively on $V = \{\mathbf{i}: i = 1, 2,..., n/2\}$.

Call $x \in D_i$ *special*, if x leaves invariant at least 2 elements of V and say that $\mathbf{i}$ is the *center* of x.

LEMMA 1. *The special elements in G form a conjugacy class B of involutions. Each element of V other than i is fixed by a unique special involution in D_i.*

Proof. Suppose $x, y \in D_i$ each fix $\mathbf{j} \neq \mathbf{i}$. Then $xy \in D_i$ fixes j and by the hypothesis that D_i acts sharply transitively on the long G_i-orbit, $xy = 1$. This shows that all special elements are involutions and establishes the last sentence of the lemma. Since a special involution is uniquely determined by its center and any other element of V that it fixes, the 2-transitivity of G on V implies that any 2 special involutions are conjugate.

PROPOSITION 1. *Let $I \subseteq V \times B$ be given by $I(x) = \{v \mid v \in V$ and $v^x = v\}$. Then I is a design with parameters $b = nt/2$, $v = n/2$, $r = t(f + 1)$, $k = f + 1$, $\lambda = f + 1$, where $t = (n - 2)/2f$ is the number of special involutions in D_1.*

Proof. Since I is G-invariant, G acts transitively on B and doubly transitively on V, I is a design. Each of the t special involutions in D_1 fixes $f = (n - 2)/2t$ elements of $V \backslash \{\mathbf{1}\}$, so $k = f + 1$. The "point" $\mathbf{1}$ is fixed by exactly one special involution with center $\mathbf{i}$ for each $\mathbf{i} \neq \mathbf{1}$, so $r = t + (n - 2)/2 = t(f + 1)$. The remaining values follow from $v = n/2$ and the familiar relations $bk = vr$ and $\lambda(v - 1) = r(k - 1)$.

LEMMA 2. *Suppose $x, y \in B$ have distinct centers. Then $xy = yx$ if and only if $| I(x) \cap I(y)| \geqslant 2$.*

Proof. If $xy = yx$, then x leaves invariant the center of y, and y leaves invariant the center of x. Conversely, suppose $| I(x) \cap I(y)| \geqslant 2$. If the center of x is in $I(x) \cap I(y)$, then $y^{-1}xy$ has the same center as x and leaves $I(x) \cap I(y)$ invariant, so Lemma 1 implies that $y^{-1}xy = x$. If neither the center of x nor the center of y is in $I(x) \cap I(y)$, pick $\mathbf{i} \in I(x) \cap I(y)$. Then x and y normalize D_i and there is $d \in D_i$ mapping the center of x to the center of y, so $\langle x, D_i\rangle = \langle y, D_i\rangle$ contains D_i as a subgroup of index 2. Thus $xy \in D_i$ and xy fixes all points in elements of $I(x) \cap I(y)$. This implies $xy = 1$.

PROPOSITION 2. *In the design I, either $\lambda = 2$, or $\lambda = 3$ and then I may be obtained from a Steiner triple system by replicating each block 3 times.*

Proof. Choose $\mathbf{i} \neq \mathbf{j} \in V$ and let x, y, z be distinct elements in $(\mathbf{i})\, I \cap (\mathbf{j})\, I$. By Lemma 1, x, y, z have distinct centers and by Lemma 2, $x^{-1}yx = y = z^{-1}yz$. Both x and y interchange the points in the center $\mathbf{k}$ of z so, as above $\langle x, D_k\rangle = \langle y, D_k\rangle$. Since both x and y fix $\mathbf{i}$ and $\mathbf{j}$, xy is a special involution in D_k. By Lemma 1, $xy = z$. Thus $\lambda \leqslant 3$. In case $\lambda = 3$ we have $I(z) = \{\text{center } x, \text{center } y, \text{center } z\}$. But $k = \lambda = 3$ so $I(z) = I(x) = I(y)$. By Lemma 2, $\lambda \geqslant 2$, so our proof is complete.

Proof of Theorem 1. Suppose the design I has $\lambda = 2$. Then a special involution $x \in D_1$ leaves invariant only one other element $\mathbf{k}$ of V, and so $(n - 2)/2$ is odd. If x interchanges $\mathbf{i}, \mathbf{j} \in V$, then the restriction of x to $\{i, i + n/2, j, j + n/2\}$ is an even permutation. Thus x is an odd permutation on $\{1,..., n\}$. Hence $D_1 \cap A_n$ (A_n is the alternating group on n letters) has odd order and is of index 2 in D_1. Thus $D_1 \cap A_n$ contains no special elements and so $D_1 \cap A_n$ acts sharply transitively on $V \backslash \{\mathbf{1}\}$. By Gorenstein [2, 10.1.4], x acts as a fixed point free automorphism of D_1, $D_1 \cap A_n$ is Abelian, and $x^{-1}\, dx = d^{-1}$ for each $d \in D_1 \cap A_n$.

If y is a special involution normalizing D_1 with center $\mathbf{k}$, then y also inverts $D_1 \cap A_n$. Thus xy is an involution, by Lemma 2, and centralizes $D_1 \cap A_1$. It follows that xy acts trivially on V and is in $Z(G)$.

Suppose that I is a nondegenerate Steiner triple system I in which each block is replicated 3 times. Then, from Proposition 2, each block is fixed pointwise by a Klein 4 group, each of whose involutions necessarily fixes no other element of V. By Hall [3. Theorem 3.2] each triangle in I generates a Fano plane. It follows directly (see, for example Hall's proof of Theorem 4.2 [3]) that I is the incidence structure of points and lines in a projective space P over the Galois field with 2 elements. Since B is a G-orbit and each special involution in G fixes exactly 3 elements of V, by a result of J. D. King [5, Theorem 2] either P is a plane and $G \simeq L_3(2)$ or P has dimension 3 and $G \simeq A_7$ or $L_4(2)$. Of these 3 groups only the first satisfies the hypothesis of our theorem and it satisfies (a) of the conclusion.

If I is a degenerate Steiner triple system (just one line), then $n = 6$.

2. Finite Möbius Planes of Hering Type I.3

In this section we prove the following theorem:

THEOREM 2. *There are no finite Möbius planes of Hering type* I.3 *and only the Miquelian planes of order* 3 *and* 5 *admit a collineation group of this type.*

We assume only the basic theory of Möbius planes found in Dembowski [1, Chap. 6].

Let M be a finite Möbius plane of order $m > 5$ possessing a collineation group of Hering type I.3. Let k be the distinguished circle with a pairing σ on the points of k so that the group is (x, x^σ)-transitive for just these pairs. Let D_x denote the group of all (x, x^σ)-dilatations and let $G = \langle D_x \mid x \in k \rangle$.

Since G is a group of Hering type I.3, the pairs $\{x, x^\sigma\}$ are sets of imprimitivity for the action of G on k. This action satisfies the hypothesis of Theorem 1. We show that this action is faithful and that G has trivial center and that the necessary orbit structure of G on $M\backslash k$ can only occur if $m = 3$ or 5 in which case M is necessarily Miquelian.

LEMMA 3. *The G orbits on $M\backslash k$ have size $\frac{1}{2}(m-1)^2$, $\frac{1}{2}(m^2-1)$ or $m^2 - m$.*

Proof. If $a \notin k$ and $x \in k$, then D_x maps a to each of the $m - 1$ points on $a\, xx^\sigma$ different from x, x^σ. If $y \notin \{x, x^\sigma\}$, then D_y maps a to points of at least $(m-1)/2$ distinct circles in $[x, x^\sigma]$. Thus $(m-1)^2/2 \leqslant |a^G|$ and $(m-1) \mid |a^G|$. Since $m > 5$, $3(m-1)^2/2 > |M\backslash k|$, so G has at most 2 orbits on $M\backslash(k)$. The result follows.

LEMMA 4. *Each element of G is an even permutation.*

Proof. Since G is generated by dilatations, it suffices to show each $g \in D_x$ is an even permutation. Suppose $g \in D_x$ has order n. Then g fixes x, x^σ, all circles in $[x, x^\sigma]$ and acts semiregularly on the remaining points of each such circle. Thus g has $(m+1)((m-1)/n)$ n-cycles in its cycle decomposition and fixes the remaining two points of M. Since m is odd, $(m+1)((m-1)/n)$ is even.

LEMMA 5. *If $1 \neq g \in Z(G)$ then either g or g^2 is the inversion with axis k.*

Proof. The set F of all fixed points of g is a union of G-orbits by [6, Theorem 10.3]. If $d \in D_x$, $x \in k$, then $x^{gd} = x^{dg} = x^g$ so d fixes x^g. Thus $x^g \in \{x, x^\sigma\}$ and g^2 fixes every point of k. If g^2 is not the inversion with axis k, then $g^2 = 1$. If $F = \varnothing$, g has $(m^2+1)/2$ 2-cycles, contrary to Lemma 4. Lemma 3 and [1, 4, p. 269] imply that F is the set of points on k, so the conclusion follows from [1.8a, p. 259].

LEMMA 6. *G does not contain the inversion with axis k.*

Proof. Suppose $g \in G$ is the inversion on k. Then g acts on the affine plane M_{x^σ} as an affine homology. Since m is odd there is an involution $d \in D_x$. The collineation dg is also an affine homology of M_{x^σ}, by [1, p. 269 and p. 272]. Since g and dg commute the axis of dg is the line l joining x to the ideal point which is the center of g, and the center of dx is the ideal point on k. Since there is a unique inversion of M with a given axis, there is a unique involution in D_x. This involution necessarily fixes each pair $\{y, y^\sigma\}$ on k. Thus if $d \in D_x$, $h \in D_y$ are the involutions in D_x, D_y, respectively, dh is an involution by Lemma 2. But dh is a dilatation [1, p. 272] and has exactly two fixed points. Since dh fixes all points of k except x, x^σ, y, y^σ there are at most 6 points on k so $m \leqslant 5$.

Proof of Theorem 2. By Lemma 6, G acts faithfully on k and $Z(G) = 1$ by Lemma 5. Therefore, conclusion (a) of Theorem 1 holds and $m = 13$. The length of a G-orbit of points must divide $|G|$, so G must have two point orbits on $M \backslash k$ and $72 = (m-1)^2/2 \mid |G| = m^2 - 1 = 168$ by Lemma 3. This contradiction implies that a finite Möbius plane admitting a collineation group of Hering type I.3 has order $m = 3$ or 5.

In case $m = 3$, M is Miquelian by [1, p. 273] and G is a dihedral group of order 8. Any such group in $\mathrm{PSL}_2(9) \leqslant \mathrm{Aut}(M)$ is of Hering type I.3.

In case $m = 5$, Krier [6] has shown that M must be Miquelian. Represent M as the points of an elliptic quadric in projective 3-space over the Galois field with 5 elements. Let k be a circle of M. Then the group H of collineations of M leaving k invariant acts on the plane cutting M in k as the orthogonal group $0_3(5) \simeq \mathrm{PGL}_2(5) \simeq S_5$. Any subgroup of H that is isomorphic to the symmetric group on 4 letters if of Hering type I.3.

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